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13. ABSTRACT (Maximum 200 words)

If the tangential component of velocity in the boundary layer becomes negative, the layer will thicken and in a short distance the assumptions leading to the boundary layer equations are no longer satisfied. One says the boundary layer has "separated". If separation occurs a global description of the flow along the lines of the iteration alluded to above is much more difficult. In this proposal they will attempt to deal with the problem. They combine a boundary layer calculation with a free streamline wake flow for the outer potential flow. In the iteration, a separation point is determined by the boundary layer equations, a wake flow is computed and the boundary layer equations are solved again with new pressure distribution on the unseparated part of the boundary. The wake flow uses Tulin's double spiral vortices to terminate a near wake and connect with free streamlines which come together at infinity downseam. This model was proposed and developed in the work; a version with improved numerical implementation. The body is approximated by a polygon, and Schwarz-Christoffel methods are used; the computational work is in computing Schwarz-Christoffel parameters and once this is done the solution is given by an analytic expression.

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Final Report for AFOSR 89-0323

Work has continued on analysis of models of fluid dynamic phenomena. A number of specific examples are given in what follows. In some of these work is in progress.

Section 1. Applications of free streamline modeling.

a) Interaction with viscous boundary layers.

The steady flow past a lifting surface at high Reynolds numbers and modest angle of attack can be thought of as consisting of two parts, the exterior flow in which the flow is essentially inviscid, and a thin region near the body in which viscosity is important, the "boundary layer." In this boundary layer special assumptions about the nature of the flow and the thinness of the domain allow approximations which simplify the equations to be solved considerably. If the simplified equations have been solved the mathematical problem becomes one of matching the solution with an external potential flow. These ideas go back to Prandtl [P]. Over the years the analysis of the boundary layer equations has been refined by many workers. In particular for many two dimensional problems averaged equations which lead to a system of ordinary differential equations for averaged quantities have been given. The inclusion of transition from laminar to turbulent flow, and the continuation of a turbulent boundary layer can also be included if certain empirical relations are accepted. The primary quantities solved for in these equations are the momentum thickness and displacement thickness; these quantities may be thought of as describing the development of the boundary layer downstream along the body. (A description of one version of these equations, is be given below. General references to this subject are [Sch] and [M], chapter 5.)

The body plus the boundary layer "strip" may be thought of as a domain complementing a potential flow region. Since the boundary layer equations require the pressure distribution on the surface (from an exterior potential flow) as data, an iteration is suggested in which the original body is augmented by the displacement thickness, a new potential flow is computed, the solution of the boundary layer equations with the new pressure distribution is obtained, and so on. In fact, various version of this have been in long use in the aerodynamics community.

If the tangential component of velocity in the boundary layer becomes negative, the layer will thicken and in a short distance the assumptions leading to the boundary layer equations are no longer satisfied. One says the boundary layer has "separated." If separation occurs a global description of the flow along the lines of the iteration alluded to above is much more difficult. In this proposal we will attempt to deal with this problem. We combine a boundary layer calculation with a free streamline wake flow for the outer potential flow. In the iteration, a separation point is determined by the boundary layer equations, a wake flow is computed and the boundary layer equations are solved again with the new pressure distribution on the unseparated part of the boundary. The wake flow uses Tulin's double spiral vortices [illegible] to terminate a near wake and connect with free streamlines which

come together at infinity downseam. This model was proposed and developed in ~~[BEI]~~ our work; a version with improved numerical implementation is described below. The body is approximated by a polygon, and Schwarz-Christoffel methods ~~[Tre], [ET]~~ are used; the computational work is in computing Schwarz-Christoffel parameters and once this is done the solution is given by an analytic expression.

It was shown in [ET] by Elcrat and L.N. Trefethen that, in the classical model of a wake originally formulated by Helmholtz [H] and Kirchhoff [K], the flow past a polygonal obstacle can be given in terms of a single analytic formula. Further, the disposable parameters can be computed reliably and efficiently using an adaptation of the earlier work of Trefethen [Tre] on the Schwarz-Christoffel transformation. If $\zeta = u - iv$ where the velocity components are u, v, and an upper half t-plane is used as a parameter domain, the velocity in the Helmholtz-Kirchhoff flow can be written

$$\zeta_0^{-1} = e^{i\gamma_n} \prod_* h_k(t)$$

where

$$h_k = \left((t - t_k)/\ (1 - t\, t_k - \sqrt{1 - t_k^2}\ \sqrt{1 - t^2}\ )\right)^{\beta_k}.$$

In this formula $\beta_k = \gamma_{k+1} - \gamma_k$ where γ_k is the angle the kth side makes with the horizontal, t_k is the parameter value corresponding to the vertex z_k, and $*$ denotes the set $t_1, \ldots, t_{n-1}, t_*$, where t_* corresponds to the stagnation point z_*.

The formula for the flow is

$$\frac{dz}{dt} = \zeta_0^{-1} \frac{dw}{dt}$$

where $w = \frac{1}{2}\ W(t - t_*)^2$ is the complex velocity potential, and W is a positive constant.

The parameter t_* is determined in terms of $t_1, \ldots, t_{n-1}$ by $\zeta_0(\infty) = 1$:

$$t_* = -cos\left(\gamma_n \pi - \sum_{k=1}^{n-1} \beta_k\ arcos\ (-x_k)\right).$$

A system of equations for W, $t_1, \ldots, t_{n-1}$ is obtained by imposing the side length conditions; the side lengths are computed using compound Gauss-Jacobi quadrature as in [ET]. The equations can be solved, in a fast and efficient manner for non-pathological geometries with a modest number of sides (up to 30 in our experience) using a Newton type method with the equally spaced values as an initial guess. (In [Tre] and [ET] the routine NSO1A of Powell [PO] was used; in related work Dias [DET], [DI] used the IMSL routine ZXSSQ.)

The Helmholtz-Kirchhoff wake model suffers from the disadvantage that the constant pressure in the "deadwater" region between the free streamlines must equal the free stream pressure; in experimentally realized flows the pressure may be nearly constant but is usually lower than the free stream pressure. Unfortunately, it is mathematically impossible to have free streamlines that join together at a finite point with a constant pressure unless it is larger than the free stream value. (This is the Brillouin paradox; see Gilbarg [G] for example.) This has lead to a number of attempts to

"close" the wake with some artificial device in order to have an underpressure and get a reasonable description of the flow near the body. (Several of these are discussed in [G]; see also [BZ], [GU].) We are interested here in one of these in particular, the double spiral vortex model of Tulin [T]. In this model there is a jump discontinuity in the pressure up to the free stream value at some point downstream on each free streamline. By Bernoulli's equation this is equivalent to a jump in speed from a value q to the free stream value which we take to be 1.

By necessity the argument of the velocity must become infinite at such a point and the left and right branches spiral around each other infinitely many times in approaching each other, hence the name given to the model. Tulin used these ideas for a linearized wake model, but "full nonlinear" versions have been implemented. (See [Fu] for example.) A version of this was given for polygonal obstacles in [BE1]. We introduce the parameters $t_s > 1$ and $t_{s'} < -1$ and the functions

$$X_s(t) = \left((t-t_s)/(tt_s - 1 + \sqrt{t_s^2-1}\ \sqrt{t^2-1}\ )\right)^{-ln\ q/\pi}$$

and

$$X_{s'}(t) = \left((1 - tt_{s'} + \sqrt{t_{s'}^2-1}\ \sqrt{t^2-1} \Big/ (t-t_{s'})\right)^{-ln\ q/\pi}$$

Then $\zeta = \zeta_0\ X_s\ X_{s'}$ will be the complex velocity. The equation for t_* is replaced by a new expression involving t_s, $t_{s'}$ also. In order to have a uniquely determined flow two additional conditions are needed to determine t_s, $t_{s'}$. In [BE1] we used two algebraic equations involving the parameters which were derived from the condition that the asymptotic width of the far wake was zero. A number of successful computations were done in the work leading to [BE1], but the side length conditions coupled with these algebraic equations are a poorly conditioned system and the computational effort was much greater than that in [ET]. This has led us to reformulate the computational problem as follows. Suppose that z_s and $z_{s'}$ are the spiral points. Then we require that

$$Real\,(z_s - z_{s'}) = 0,$$

(the near wake has a symmetric closure), and that

$$Im\ (z(T) - z(-T)) = 0$$

for some "large" value of T. This second condition is a finite approximation to the condition of zero asymptotic width for the far wake. (In an airfoil problem a value of $T=4$ typically puts $z(T)$ about 20 chord lengths downstream.) These are then two additional conditions and the count is right in determining $W, t_1, \ldots, t_{n-1}, t_s, t_{s'}$. The new equations involve integrals which are evaluated in essentially the same way as the others, and the system of equations obtained is well behaved. When both methods work (the present method and the one presented in [BE1]), the results differ by less than a few percent.

We give here the boundary layer equations in the form that we will use then.

The motion of a viscous fluid flowing in a thin layer near a solid surface can be described by the

equations

$$uu_x + vu_y = -p_x + R^{-1} u_{yy}$$
$$u_x + v_y = 0$$

where $u = v = 0$ on $y = 0$, (u, v) approaches $(U(x), 0)$ as y goes to infinity, and

$$p(x) + \frac{1}{2} U^2(x) = \text{constant}.$$

The equations have been written in dimensionless form and R is the Reynolds number; curvature of the surface has been neglected. The solutions of these equations are an asymptotic limit of solutions of the Navier-Stokes equations which are to be matched with an outer potential flow which has velocity distribution $U(x)$ on the edge of the boundary layer. In particular, the solutions may be desired downstream of a stagnation point. Then a useful approximation can be obtained by averaging to obtain ordinary differential equations for the displacement thickness,

$$\delta(x) = \int_0^\infty (1 - \frac{u}{U})\, dy$$

and momentum thickness,

$$\theta(x) = \int_0^\infty \frac{u}{U} (1 - \frac{u}{U})\, dy .$$

If the flow is turbulent one set of such equations are

$$\theta' + (H+2)\,(U'/U)\theta = \frac{1}{2} c_f ,$$

(the momentum integral equation), and

$$\frac{1}{U} (U\theta H_1)' = .0306\, (H_1 - 3)^{-.6169}$$

where $H=\delta/\theta$ is the "shape factor" and c_f is the turbulent skin friction. We will take c_f to be given by the Ludwieg-Tillman expression

$$c_f = .246\ \times 10^{-.678H}\, (RU\theta)^{-.268}.$$

H_1 is an (empirically derived) function of H(cf. Moran [M]).

In the initial, laminar, flow region we use the method of Thwaites. Here an empirical correlation between H and the skin friction allows exact integration of the momentum integral equation. Transition is determined by Michel's method, and after transition the above equations (Head's method) are used. Our implementation is essentially the program INTGRL given by Moran [M]. This code is used as a subroutine which integrates downstream from a stagnation point until separation occurs (either laminar or turbulent). In the turbulent region separation is taken to have occurred when H reaches a certain value. In INTGRL this value is 2.4; we have used the value 1.8. The value we have taken was observed in one set of experiments for a GA(W)-2 airfoil that we discuss below, and this choice led to excellent agreement with these experiments.

We will describe now in more detail the iteration that we have in mind for separated flows. Suppose that at a given angle of attack a potential flow (without separation) has been obtained for the

obstacle under consideration, and that the pressure distribution is known as a function of arc length downstream from the stagnation point. (We are tacitly assuming an airfoil section at positive angle of attack so that only separation on the top is investigated. This is not essential and the generalization of this procedure to other situations is straightforward). This is then used as data, together with the Reynolds number, for a boundary layer calculation, the output of which is the separation point, if any, and θ and H as functions of arc length up to separation. If separation occurs a wake flow is computed as described in section 1 using the separation point and separation pressure from the boundary layer calculation as data. The displacement thickness is added to the current profile. This provides a new pressure distribution which can be used as input to the boundary layer subroutine. The process is iterated until the separation point does not move further upstream.

We report here the results of some numerical calculations and make comparisons with experiments.

In these examples we have used as input the pressure distribution for attached flows. Typically convergence was obtained for a given angle of attack in 2-3 iterations with a total CPU time of less than 2 minutes on the IBM 3 ES-9000, model 440 at Wichita State University.

The experiments of Wentz, Seetheram, and Rogers reported in [WSR] were our principal benchmark. These were carried out at a Reynolds number of 2.2 million, and we show in figures 1-3 below our computations plotted with experimental values for several angles of attack for a GAW-2 airfoil.

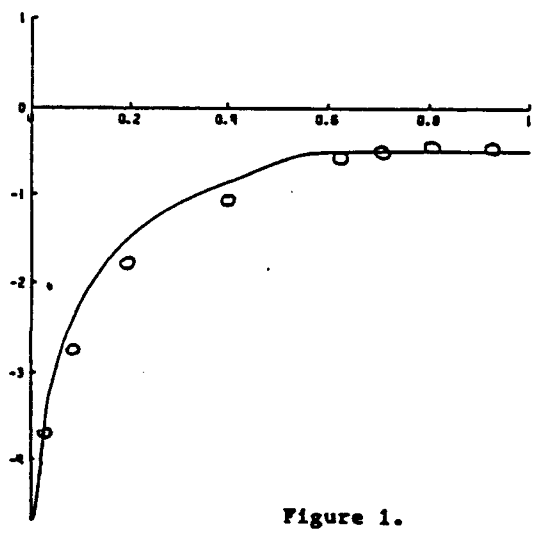


Figure 1.

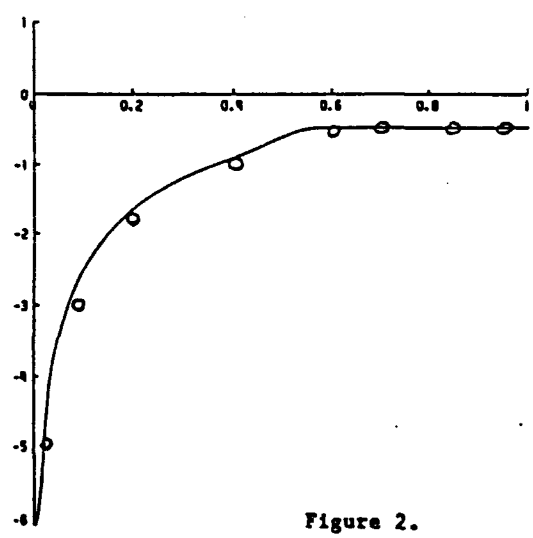


Figure 2.

In addition we have run the case α=18°, R= 4.3×10^6 and compared with the experimental curve given in [MB]. The result is shown below, in figure 4..

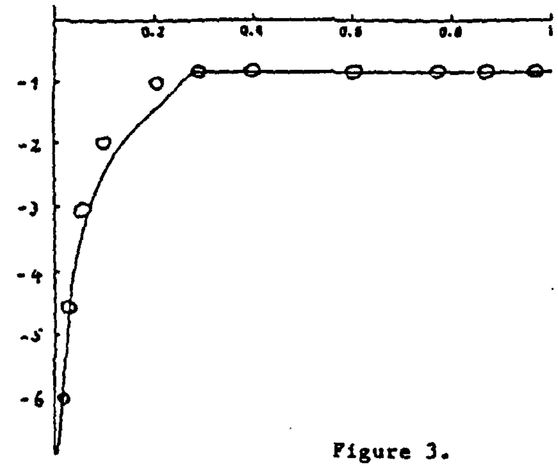


Figure 3.

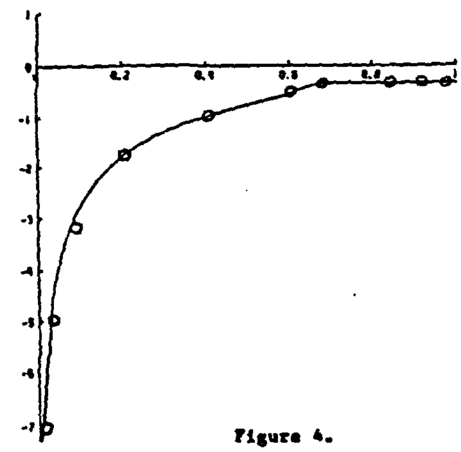


Figure 4.

In figure 5 we show a typical flow net.

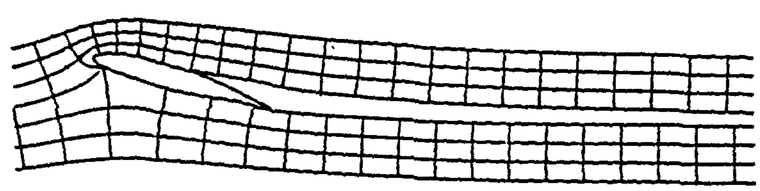


Figure 5.

A major impetus for this work is its use in matching for 3-D flows as in [BE2]. Work is in progress on doing this when separation on a section is determined by the Reynolds number. The matching leads to a fixed point equation relating the parameters in the inner flow on a two dimensional cross section to the circulation and source strength in a lifting-line-source model of the outer solution. The velocity potential of the outer flow has an inner expansion

$$\phi = x - y\,\frac{\varepsilon}{4\pi}\int f_1'(t)\,\frac{dt}{z-t} - \frac{1}{2\pi}\,f_1(z)\ arctan\ (y/x) + \frac{1}{2\pi}\,f_2(z)\ ln\ r + o(\varepsilon)$$

where z is transverse coordinate, ε is the aspect ration, and the integral is a Cauchy principal value. The inner solutions has an outer expansion

$$\Phi = x - \alpha(z)\,y + B(z)\ lnr - C(z)\ arctan\ (y/x) + o(\varepsilon)$$

where α is the downwash. Matching to first order in ε requires

$$f_1(z) = 2\pi C(z),\ f_2(z) = 2\,\pi B(z).$$

The downwash is then given by

$$\alpha(z) = \frac{\varepsilon}{2\pi}\int f_1'(t)\,\frac{dt}{z-t}\,.$$

$B(z)$, $C(z)$ are functions of the parameters in the wake flow, and this yields a fixed point equation which can be solved numerically as in [BE2].

The separation point and pressure on a section are determined as described previously, so that the model obtained of the 3-D wake contains cross flow pressure gradients and is not a constant pressure region as in [BE2]. We can formulate a crude model of the flow in the 3-D wake with changes in the sign and magnitude of $B(z)$ (this is essentially the source strength); there is some hope that we may be able to use this to help explain the recirculation cells in the wake behind a wide, bluff body. Work is in progress on this.

b) Mixed design-analysis for 2-D airfoils.

In the work described above for separated flow on an airfoil it was found that a polygonal approximation to an airfoil provides surprisingly good approximations for pressure distributions. This requires some explanation.

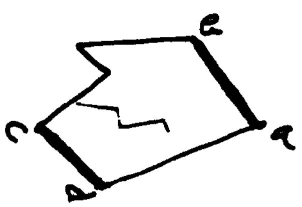
The exact velocity profile (and hence pressure distribution) of a flow past a polygon has infinities (or zeros) at each vertex. It is extremely ragged, nonmonotone and a poor approximation to that of a smooth body. On the other hand if pressures are taken at mid-points of straight boundary segments and smoothly interpolated this difficulty disappears. In fact this is closely related to common practice with panel methods in aerodynamics. Our recent experience has been that a Schwarz-Christoffel solution does surprisingly well with a relatively small number of sides ("panels").

Since a wake model is nothing but a replacement of flow tangency with a specification of the pressure over a part of the flow, the suggestion is natural that this can be done in a more general way than simply requiring the pressure to be constant on the wake streamlines. To specify pressures over a part of the airfoil or wake boundary leads to a "generalized parameter problem" for a Schwarz-Christoffel transformation, as described in [Tr RD]. An interesting flow problem which can be used as a first pass in testing the feasibility of this idea is the formation of a "leading edge separation bubble" as depicted below.

We will begin by investigating this problem.

c) Crack detection.

Another problem that we are working on is described below. This is not a fluids problem. It is placed here in the discussion because of the methods to be used.



Consider a polygon P with four distinguished vertices a, b, c, d. If fixed constant voltages are imposed on the arcs ab and cd and the rest of the boundary is insulated, the resistance of this polygonal "circuit element" can easily be measured experimentally or computed using the solution of a mixed boundary vlaue problem for Laplace's equation. The computation has been give a particularly elegant formulation by Trefethen in [TrRD] using the Schwarz-Christoffel transformation. Suppose now that a crack of unknown (polygonal) shape extends into P. If this crack is described by N geometrical parameters then an algorithm for their determination can be given as follows. Suppose that the "terminals" ab and cd are moved around the boundary N times to give N different values for the resistance. These measurements then give N conditions with which to determine the geometry of the crack. Trefethen did essentially this in the case $N=1$ although he considered the crack length as a variable to be tuned in order to obtain a prescribed resistance. The determination of several geometrical parameters is another "generalized parameter problem" for a Schwarz-Christoffel map and it seems very likely that this proposed algorithm can be made to work.

A similar idea can be put forward if the crack is interior to the domain. The physics is the same, but a doubly connected version of the polygonal mapping problem must be used. Fortunately, the computational details of this have been worked out by Daeppen [Daep]. The program that he wrote is not available, but we plan to write our own program based on his ideas as a part of this project. For a three dimensional problem conformal mapping is not available and a computational algorithm will be more difficult, but is seems likely that the two dimensional problem will serve as a guide for what is possible.

Section 2. Variational Methods for Vortex Flows.

a) Steady Flows.

The use of variational methods for the study of steady, planar flows of an ideal fluid with regions of nonzero vorticity has been in progress since the time of Kelvin [K]. More recently, Benjamin [B76], Turkington and Friedman [FT], and Burton [BVR] have studied vortex rings using this approach. The planar problem was studied by Turkington [T83], Burton [B], and Elcrat and Miller [EM1], [EM2]. Numerical results have been given in [EM3] and [TE]. The work of [B] is of particular importance for this proposal in that the vorticity in the "core" regions need not be constant; the class in which variational solutions are found is the set of rearrangements of a fixed function (ie. functions whose level sets have the same measure) and the profile function which relates the vorticity to the stream function is determined as a part of the solution. (This is in line with Benjamin's original suggestions in [B76].) One difficulty with Burton's theory is that it does not always apply directly to situations of physical interest. We plan to extend his results to flow in a bounded domain with inflow and outflow boundary conditions on parts of the boundary. This can be used to model flow in a channel as well as

approximating the flow past an obstacle in an unbounded domain. As a part of our efforts we plan to develop an iterative scheme for the solution in which the profile function and hence the vorticity are determined iteratively using variational methods. After successful completion of this task, a numerical solution of the problem can be obtained in a straightforward way. In a related work axisymmetric flows past symmetric bodies will be studied.

These flow problems are relevant to high Reynolds number flows past bluff bodies and in the interior regions such as an air duct. In practical situations the flow in these configurations is likely to be unsteady and turbulent, but steady flows can be useful in approximating averaged quantities; also the possibilty of introducing external controls which maintain the stability of a steady flow is a Holy Grail for fluid dynamicists, and further information about steady flows improves the chances for attaining this goal.

The iterative scheme mentioned above will now be explained in the context of the model problems of maximizing the energy for vortices whose support are contained in a fixed bounded domain. A maximum is sought for

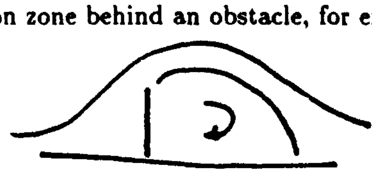
$$E(\omega) = \frac{1}{2} \int_{\Omega} \omega K \omega$$

where K is the Green's operator for the negative Laplacian in Ω, and the maximum is sought in the class of functions which are rearrangements (level sets have the same measure) of a fixed function in $L^{p}(\Omega)$. (If the stream function $\psi = K\omega$ is introduced, an integration by parts shows that $E(\omega)$ is the Dirichlet integral of ψ). The iteration that we will work with is based on ideas given by Eydeland and Turkington [EyTu]. A sequence converging to a maximizer is generated by

$$\omega^{j} \ argmax \ \frac{1}{2} \int_{\Omega} \omega K \omega^{j-1}.$$

We will take the maximum over the class of rearrangments defining the original variational problem. We have proven that this sequence converges in $L^{p}(\Omega)$ and, further, that there are increasing profile functions ϕ_j, such that $\phi_j\ (\psi^j) = \omega^j, \psi^j = K\omega^{j-1}$, and that the sequence ϕ_j converges almost everywhere. We plan to give a numerical implementation of this iteration by approximating the rearrangement condition at a finite number of values.

There is another class of variational problems for 2-D vortex flows that we have considered. The physics of these flows can be associated with large Reynold's number limits for steady solutions of the Navier Stokes equations in the flow past a bluff body. (See, for example, [S], in particular the discussion of "Prandtl-Bachelor" flows.) They can also be thought of as having practical value as averages for turbulent flows. The problem is to find a flow which contains a vorticial region in the separation zone behind an obstacle, for example as in

 ,  .

The variational problem for such flows has some formal similarity to that discussed above, but is different in subtle ways. In particular, if the vorticity is constant, say equal to -ω, then a functional which is associated with this problem is given by

$$\mathrm{I} = \frac{1}{2} \int_D |\nabla \psi|^2 + \omega \int_B \psi$$

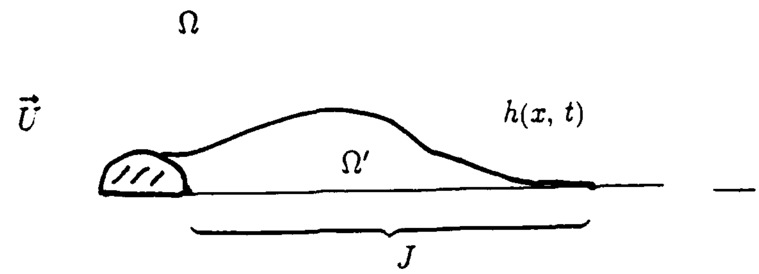
where ψ is the stream function, D is a "control volume" in which the flow takes place, and B is the vorticial region. In fact, a number of interesting flows were computed by an interation which minimizes this functional by M.A. Goldshtik several years ago [Gold]. (This work does not appear to be widely known in the west. We learned about it during a recent visit by Goldshtik to Wichita.) A surprising fact about this functional is that a <u>maximizer</u> B, if it existed, could not have interior points. The functional that was considered by Turkington in [T83], for example, can be brought into the above form; in that work a <u>maximizer</u> was found subject to a constraint on the area (circulation) of the vortex patch.

We plan to pursue further work along these lines using the ideas of Goldshtik. In particular a "vortex sheet" or jump in the speed across the boundary of the vortex region arises if the term $\sigma|B|$ is subtracted from I; σ is the magnitude of the jump. We have given a theoretical treatment for a model problem in which a vorticial region bounded by a vortex sheet is contained in a bounded domain. The minimizer is bounded by a Lipschitz curve, and an essential part of the analysis is to establish Hadamard type variational formulas (Garabedian, PDE, chatper 15) for Lipschitz domains. We hope to give a similar treatment to problems of the type considered by Goldshtik.

b) <u>Hamiltonian formulation of time dependent wakes.</u>

Consider a two dimensional flow in which there are bounded regions in which the fluid motion has constant vorticity. The flow can be described by harmonic velocity potentials in the various regions; these are perturbations from a fixed constant vorticity flow in the rotational regions. A complete description of the flow requires these time dependent potentials and the boundaries of the rotational regions.

To fix ideas consider a symmetric flow past a symmetric body which is uniform at infinity and has a single vortex region in the upper half plane adjacent to the body. The situation is depicted below.



In Ω the velocity is given by $\nabla\phi + U\underline{i}$ and in Ω' by $\nabla\phi' + \frac{\omega}{2}(-y, x) + U\underline{i} = (\psi_y, -\psi_x)$. In recent work we have shown that this problem can be given a Hamiltonian formulation. This uses ideas recently developed by Benjamin [BeBr], [Be], [BWW]. If K is the perturbation energy in the flow then

$$K = \frac{1}{2}\int_\Omega |\nabla\phi|^2 + \frac{1}{2}\int_{\Omega'}(|\nabla\phi'|^2 + \frac{\omega}{2}(-y, x)\cdot\nabla\phi' + f)$$
$$-U\int_J(\Phi - \Phi')h_x$$

where f is a fixed function, we have taken the density of the fluid to be 1, and $\Phi = \phi(x, h(x,t), t)$ $\Phi' = \phi'(x, h(x, t), t)$. We denote by $\delta K/\delta u$ a variational derivative with respect to a quantity u as in [BeBr], [BWW].

Let $\zeta = \Phi - \Phi'$. We have shown then that

$$h_t = \frac{\delta K}{\delta\zeta}, \; \zeta_t = -\frac{\delta K}{\delta h}.$$

These express kinematical compatibility and the Bernoulli equations in Ω, Ω'. We plan to extend this to nonsymmetric flows, multiple vortex configurations, and axisymmetric flows.

There are several important features of this new formulation of the unsteady wake problem. The evolution is described in terms of quantities defined only on the boundary of the vorticial region bounded by vortex sheets, but the formulation is not inherently singular as is the case, for example, with the Birkoff-Rott equation for evolution of vortex sheets; a desingularization should not be necessary for a numerical solution as in [Kr]. Accurate and efficient computation of Dirichlet Integrals of harmonic functions is required, of course, but we have extensive experience in doing this, and work is in progress in further developing this capability (cf. discussion of conformal mapping below.)

The Hamiltonian formulation leads naturally to variational principles for steady flows. In particular, the computation of Prandtl-Bachelor flows and the establishment of their relevance for large Reynolds number viscous flows is a long standing problem ([S]), and the new variational principles that arise from this work may be of help in resolving this.

If the wake problem is idealized by removing the obstacle (eg. it is far upstream and its effects are contained in the vorticity distribution), a flow problem which has much in common with the water wave problem is obtained (cf.[BWW].) Then not only steady solutions, but steadily translating

configurations can be studied. This is relevant to unsteady oscillations in the wake behind a bluff body.

c) Conformal Mapping

The conformal map of a canonical domain onto one with an irregular boundary is a very useful tool in solving partial differential equation problems involving harmonic functions. If the canonical domain is the unit disk and the curve bounding the irregular domain is smooth the methods which use Fourier series and conjugation to determine the boundary correspondence function, and hence the map, are the most widely used and successful. Among these two that compute the boundary correspondence $s=s(\theta)$, s=arc length along the target curve, using Newton's method for updating the approximation to $s(\theta)$ were developed by Fornberg [FO] and Wegmann[W]. They may be thought of as arising from a singular integral equation for $s(\theta)$ on the unit circle, and the linear equation which has to be solved at each Newton's step requires that

$$z(s_n(\theta)) + z'(s_n(\theta))\, U_n(\theta) \tag{4.1}$$

extend to an analytic function inside the disk. Here $z(s)$ is the coordinate function for the curve and $s_{n+1} = s_n + U_n$. There are some differences in detail in how this problem is solved at the discrete level in the two approaches; Wegmann solves a Riemann-Hilbert problem in the disk, whereas Fornberg projects onto functions analytic in the disk. Both methods are very effective; the nonlinear iterations usually converge from an arbitrary initial guess although known theoretical results show only local convergence. The linear equations are solved using FFT's and the solution is fast. (We have found Fornberg to be more robust; Wegmann requires fewer FFT's per Newton iteration.) The method provides as part of the solution an approximate Taylor series for the map.

A typical application was dealt with by Fornberg, namely the time evolution of a free surface under the influence of gravity. A new conformal map is computed at each time step. In this and other problems domains which are thin or "fingered" on part of their boundary often arise. When this happens the "crowding" phenomena occurs [G], [MZ] and a large number of Fourier coefficients are required. In this case it is natural to ask if use of a different canonical domain will make the problem more tractible. In the case of polygon a Schwarz-Christoffel method has been given in [HT]. Another new method for doing this is proposed here.

The difficulty with Fourier series for thin domains is that a large number of terms in a Taylor series are required for resolution, i.e. the problem is one of discretization error. We will consider domains which are "thin" in one direction and use the inside of an ellipse which is the inverse image under the Joukowski map $z=\frac{1}{2}(t+\frac{1}{t})$ of $|t| = \rho$ as our canonical domain. In place of Taylor series we use the Tchebychev series

$$\frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n T_n(z); \tag{TS}$$

$T_n(z)$ are the "Faber polynomials" for this domain and are appropriate for the expansion of analytic functions here. It can be shown that a function defined on the ellipse decomposes into the sum of a

Faber series and a function analytic on the exterior. The analogue of Fornberg's projection, in which negative index Fourier coefficients are set to zero, is therefore possible. It can be expressed in the form: integral of f times T_n equals zero for all n. It is convenient to use the Joukowski transformation and a scale change to transform this to the unit circle. If ζ is the new variable and we denote the transplanted function again by f this condition is equivalent to $a_{-n} = \rho^{-2n}\, a_n$ for the Laurent coefficients of $f(\zeta)$. Since our ellipse is conveniently parametrized by $z = \frac{1}{2}\,(\rho e^{i\theta} + \rho^{-1}\, e^{-i\theta})$ we may as well think of the equation (2.1) as being imposed on 2π periodic functions or functions defined on $\zeta = e^{i\theta}$ as convenience dictates. If P_+, P_- are the projections onto functions analytic in the interior and exterior of the circle our condition on the Laurent coefficients takes the form

$$P_- f = LP_+\, f + Jf$$

where $Jf = a_0$, and L multiplies Taylor coefficients by ρ^{-2n} and performs an inversion. If we write the right hand side of (4.1) as $z + tU$, we obtain

(4.2) $$(P_- - LP_+)\, tU = -(P_- - LP_+)\, z.$$

If this is discretized and use is made of the N – dimensional discrete Fourier transform F_N we obtain

(4.3) $$A\, F_N\, T\, U = -A\, F_N\, z$$

where T is a diagonal matrix (values of t) and A has the block structure

$$A = \begin{bmatrix} 0 & 0 \\ \hat{P} & \hat{I} \end{bmatrix},$$

the (square) submatrices having order $N/2$ with

$$\hat{P} = \begin{bmatrix} 2\rho^{-\frac{N}{2}} & 0 & -2\rho^{2-N} & \cdots & 0 \\ & & & & \rho^{-2} \\ & & & \rho^{-4} & \\ & & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & & \end{bmatrix}, \quad \hat{I} = \begin{bmatrix} \frac{1}{2} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

The first row arises from discretization of the series (TS) vanishing at the origin. This amounts to $N/2$ complex equations for the N real components of U, in complete analogy to equation ($12b$) of Fornberg [FO]. We introduce the even-odd decomposition as done there and use the Gauss decomposition.

$$F_N = \begin{bmatrix} F_M & W_1 F_M \\ F_M & W_2 F_M \end{bmatrix}$$

where $M = N/2$ and W_j are diagonal matrices with roots of unity on the diagonal. This leads to an

equation of order M:

$$(\widehat{P} + \widehat{I})\, F_n\, T_0\, U_0 + (\widehat{P}W_1 + \widehat{I}W_2)\, F_M U_1 = e$$

where e is obtained by mulitplying the matrices on the left times z, and U_0, U_1 are the even and odd components of U. Finally, the fact that U is real must be used to obtain a full system. As in [FO] an $M \times M$ system can be obtained for U_0, and U_1 is obtained by back substitution. The matrices involved are either unitary or have an explicit structure that requires only $O(M)$ operations for equation solving; a conjugate gradiant step can be done in a comparable number of operations to those in [FO]. The condition that a point is fixed on the boundary of the target curve can be used as in [FO] to compensate for the fact that $\widehat{P} + \widehat{I}$ has an eigenvalue near zero. Finally, as $\rho \to \infty$ Fornberg's method is retreived, and the eigenvalue structure of the operators in (4.2) and matrices in (4.3) can be analyzed as in [WE], [WI].

A strong motivation for obtaining this method is to deal with cases in which the vorticial domains in inviscid flow problems become elongated. In particular an efficient computation of the Dirichlet integral can be done using the transplantation to the ellipse. This may be of great value in both steady and unsteady problems in which the Hamiltonian functional involves the Dirichlet integral.

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